Dissipative Weakly Almost Periodic Functions

By JOHN F. BERGLUND

Using the Bochner-von Neumann [2] definition of the almost periodic functions on a group, Eberlein [4] analogously defined the weak almost periodic functions. He then proceeded to show that the set of weak almost periodic functions enjoys many of the properties of the set of almost periodic functions: *e.g.*, that it is a uniformly closed linear space, indeed a C*-algebra, and, when the group is locally compact and abelian, it has an invariant mean and consists of uniformly continuous functions. Furthermore, all the functions "of interest" in harmonic analysis are weak almost periodic for a locally compact abelian group, at least; *viz.*, the almost periodic functions, the functions vanishing at infinity, and the Fourier-Stieltjes transforms.

In his second paper on the subject, Eberlein [5] considered the formal Fourier series

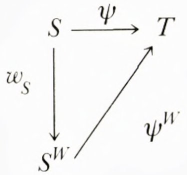
$$\sum_{\lambda \in \Gamma} a_\lambda < t, \lambda >$$

associated with a weak almost periodic function f defined on a locally compact abelian group G with dual group Γ, where, if M denotes the unique invariant mean on the set $WAP(G)$ of weak almost periodic functions on G, then

$$a_\lambda = M[f(s) <-s, \lambda>].$$

Unlike the case for almost periodic functions, the Fourier series is not uniquely associated with the weak almost periodic function f. In fact, Eberlein showed that there is a unique decomposition $f = f_1 + f_2$, where f_1 is almost periodic and has the same Fourier series as f and f_2 is weak almost periodic with $M(|f_2|^2) = 0$. We are concerned with this latter type of function, which we call *dissipative.*

A more general view of weak almost periodic functions was introduced by de Leeuw and Glicksberg in [3]. The Bochner-von Neumann definition of almost periodic is as follows: Let f be a continuous bounded complex-valued function on the topological group G. Define f_s by $f_s(t) = f(ts)$, $t \in G$, and define $O(f) = \{f_s | s \in G\}$. Then f is *almost periodic* if $O(f)$ is relatively compact in the norm topology of $C(G)$. Eberlein required that $O(f)$ be relatively compact in the weak topology to get the *weak almost periodic* functions. Clearly, not all the properties of a topological group are required for these definitions to make sense; in particular, neither group inverses nor joint continuity of the multiplication is required. Therefore de Leeuw and Glicksberg defined weak almost periodic functions on semigroups with separately continuous multiplication. They then proceeded to define and exploit the weak almost periodic compactification (w_S, S^W) of a semitopological semigroup S; that is S^W is a compact semitopological semigroup, w_S:$S \to S^W$ is a continuous homomorphism, $w_S(S)$ is dense in S^W and if $\psi : S \to T$ is a continuous homomorphism into a compact semitopological semigroup, then there is a continuous homomorphism $\psi^W : S^W \to T$ such that



commutes. A function f is weak almost periodic on S if and only if there is a continuous function $f^W \in C(S^W)$ such that $f = f^W \circ w_S$. Properties of the algebra of weak almost periodic functions are reflected as properties of the compactification. For example, $WAP(S)$ has an invariant mean if and only if S^W has a group as its minimal ideal $K(S^W)$. The dissipative functions f are, in that case, the ones for which $f^W|_{K(S^W)} \equiv 0$.

Soon after Eberlein's original paper [4], a question arose as to whether there were any functions $f \in WAP(G)$, where G is a locally compact abelian group, which were not uniform limits of Fourier-Stieltjes transforms of measures on Γ. Given the decomposition theorem, this amounts to the question of whether there are any dissipative functions on G which are not uniform limits of Fourier-Stieltjes transforms. In 1959, W. Rudin [7] gave an example of a dissipative function which cannot be approximated by Fourier-Stieltjes transforms. His example on the additive group **Z** of integers is the following:

$$f(m) = \begin{cases} e^{in\log n} & \text{if } m = k!n \; (k = 1, 2, 3, \ldots, 1 \leq n \leq k) \\ 0 & \text{otherwise.} \end{cases}$$

I am not myself interested in the Fourier-Stieltjes aspect of this problem, but in the behavior of dissipative functions.

Writing out the support of Rudin's function we have

$$\text{supp}\,(f) = \{k!n \mid k = 1, 2, 3, \ldots; 1 \leq n \leq k\}$$

$$= \{1, 2, 4, 6, 12, 18, 24, 48, 72, 96, 120, 240, \ldots.\}$$

Note that there are larger and larger gaps in this set of integers. How typical is this of dissipative weak almost periodic functions? At first glance, one must conclude that it is not very typical since every function vanishing at infinity is weak almost periodic and adding one such to f will give us a weak almost periodic function with perhaps no gaps in its support. The following theorem shows that, taking into account the functions vanishing at infinity, the above pattern is typical.

THEOREM. Let $(G,+)$ be a locally compact (not necessarily abelian) topological group, and let S be a closed, noncompact subsemigroup of G. Suppose that $WAP(S)$ has an invariant mean and that addition is continuous at every point of $S\times S^{W}$. Then the following statements about a function $f \in WAP(S)$ are equivalent:

(a) *$f^{W} \equiv 0$ on the minimal ideal $K(S^{W})$ of S^{W}.*
(b) *$M(|f|) = 0$, where M is the unique invariant mean on $WAP(S)$.*
(c) *The zero function is in the weak closure of the orbit $O(f)$.*
(d) *For every $\epsilon > 0$ and every compact subset K of S, there is an element $s \in S$ such that*

$$\epsilon > \|R_s f\|_K = \sup\{|f(k+s)| : k \in K\}.$$

This is Theorem 3.4 of [1], and the proof is given there.

Although dissipative functions f are such that $|f|$ has large gaps in its "above ϵ" support, these gaps cannot be arbitrarily far apart, as the following theorem from [1] shows:

THEOREM. Let $(G,+)$ be a locally compact topological group, and let S be a closed, noncompact, subsemigroup of G containing the identity 0. Suppose that $WAP(S)$ has an invariant mean and that addition is continuous at every point of $S\times S^{W}$. Suppose f is a dissipative weak almost periodic function on S. Let U_0 be a compact neighbourhood of the identity 0 of G and let $\epsilon > 0$. Then there is a compact neighbourhood $V = V(U_0, f, \epsilon)$ of 0 in G such that, for every $s \in S$, there exists $r \in S$ such that

$$(V+s) \cap (U_0 \cap S + r) \neq \emptyset$$

and

$$\epsilon > \|R_r f\|_{U_0} = \sup\{|f(t+r)| : t \in U_0 \cap S\}.$$

(Loosely speaking, this says that no matter where V is placed in S, it will intersect a set as big as U_0 where $|f|$ dips below ϵ;that is, these spots are relatively dense in S.)

The above theorems give us some reasonable information on the behavior of dissipative weak almost periodic functions, but a more desirable outcome would be an easily verified condition which would identify dissipative weak almost priodic functions among all bounded continuous complex-valued functions. Sufficient conditions were given by W. Rudin [7] and D. E. Ramirez [6], but they are far from necessary as has been shown by W. Ruppert [8]. Ruppert showed that functions f such as those produced by Rudin and Ramirez must vanish on $[S^W \backslash w_S(S)]^2$. On the other hand, dissipative functions need only vanish on $K(S^W)$ and, in general, $[S^W \backslash w_S(S)]^2 \neq K(S^W)$.

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